$\ddot{g}$ -Closed Sets in Topology

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Abstract **-- In this paper, we offer a new class of sets called $\ddot{g}$ -closed sets in topological spaces and we study some of its basic properties. It turns out that this class lies between the class of closed sets and the class of g-closed sets.**

Key words and Phrases: **Topological space, g-closed set, $\ddot{g}$ -closed set, $\ddot{g}$ -open set, ω -closed set.**

1. Introduction

In 1963 Levine [19] introduced the notion of semi-open sets. According to Cameron [8] this notion was Levine's most important contribution to the field of topology. The motivation behind the introduction of semi-open sets was a problem of Kelley which Levine has considered in [20], i.e., to show that cl(U) = cl(U $\cap$ D) for all open sets U and dense sets D. He proved that U is semi-open if and only if cl(U) = cl(U $\cap$ D) for all dense sets D and D is dense if and only if cl(U) = cl(U $\cap$ D) for all semi-open sets U. Since the advent of the notion of semi-open sets, many mathematicians worked on such sets and also introduced some other notions, among others, preopen sets [22], α -open sets [24] and β-open sets [1] (Andrijevic [3] called them semi-pre open sets). It has been in shown [12] recently that the notion of preopen sets and semi-open sets are important with respect to the digital plane.

Levine [18] also introduced the notion of g-closed sets and investigated its fundamental properties. This notion was shown to be productive and very useful. For example it is shown that g-closed sets can be used to characterize the extremally disconnected spaces and the submaximal spaces (see [9] and [10]). Moreover the study of g-closed sets led to some separation axioms between T_0 and T_1 which proved to be useful in computer science and digital topology (see [17] and [14])).

Recently, Bhattacharya and Lahiri [5], Arya and Nour [4], Sheik John [29] and Rajamani and Viswanathan [26] introduced sg-closed sets, gs-closed sets, ω -closed sets and αgs -closed sets respectively.

In this paper, we introduce a new class of sets namely $\ddot{g}$ -closed sets in topological spaces. This class lies between the class of closed sets and the class of g-closed sets. This class also lies between the class of closed sets and the class of ω -closed sets.

2. Preliminaries

Throughout this paper (X, τ) and (Y, σ) (or X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and Ac denote the closure of A, the interior of A and the complement of A respectively.

We recall the following definitions which are useful in the sequel.

2.1. Definition

A subset A of a space (X, τ) is called:

(i) semi-open set [19] if A $\subseteq$ cl(int(A));
(ii) preopen set [22] if A $\subseteq$ int(cl(A));
(iii) α -open set [24] if A $\subseteq$ int(cl(int(A)));
(iv) β-open set [1] (= semi-preopen [3]) if A $\subseteq$ cl(int(cl(A)));
(v) Regular open set [30] if A = int(cl(A)).

The complements of the above mentioned open sets are called their respective closed sets.

The preclosure [25] (resp. semi-closure [11], α -closure [24], semi-pre-closure [3]) of a subset A of X, denoted by pcl(A) (resp. scl(A), α cl(A), spcl(A)) is defined to be the intersection of all preclosed (resp. semi-closed, α -closed, semi-preclosed) sets of (X, τ) containing A. It is known that pcl(A) (resp. scl(A), α cl(A), spcl(A)) is a preclosed (resp. semi-closed, α -closed, semi-preclosed) set. For any subset A of an arbitrarily chosen topological space, the semi-interior [11] (resp.

α-interior [24], preinterior [25]) of A, denoted by sint(A) (resp. α int(A), pint(A)), is defined to be the union of all semi-open (resp. α-open, preopen) sets of (X, τ) contained in A.

2.2. Definition

A subset A of a space (X, τ) is called:

(i) a generalized closed (briefly g-closed) set [18] if cl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ). The complement of g-closed set is called g-open set;

(ii) a semi-generalized closed (briefly sg-closed) set [5] if scl(A) ⊆ U whenever A ⊆ U and U is semi-open in (X, τ). The complement of sg-closed set is called sg-open set;

(iii) a generalized semi-closed (briefly gs-closed) set [4] if scl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ). The complement of gs-closed set is called gs-open set;

(iv) an α-generalized closed (briefly α g-closed) set [21] if α cl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ). The complement of α g-closed set is called α g-open set;

(v) a generalized semi-preclosed (briefly gsp-closed) set [25] if spcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ). The complement of gsp-closed set is called gsp-open set;

(vi) a ĝ-closed set [31] (ω-closed [29]) if cl(A) ⊆ U whenever A ⊆ U and U is semi-open in (X, τ). The complement of ĝ-closed set is called ĝ-open set;

(vii) a αgs-closed set [26] if α cl(A) ⊆ U whenever A ⊆ U and U is semi-open in (X, τ). The complement of αgs-closed set is called αgs-open set;

(viii) ψ-closed set [23, 32] if scl(A) ⊆ U whenever A ⊆ U and U is sg-open in (X, τ). The complement of ψ-closed set is called ψ-open set;

(ix) a $\ddot{g}_\alpha$-closed set [27] if α cl(A) ⊆ U whenever A ⊆ U and U is sg-open in (X, τ). The complement of $\ddot{g}_\alpha$-closed set is called $\ddot{g}_\alpha$-open set.

2.3. Remark

The collection of all $\ddot{g}$-closed (resp. $\ddot{g}_\alpha$-closed, ω-closed, g-closed, gs-closed, gsp-closed, α g-closed, αgs-closed, sg-closed, ψ-closed, α-closed, semi-closed) sets is denoted by $\ddot{G}$ C(X) (resp. $\ddot{G}_\alpha$ C(X), ω C(X), G C(X), GS C(X), GSP C(X), αg C(X), αGS C(X), SG C(X), ψ C(X), α C(X), S C(X)).

The collection of all $\ddot{g}$-open (resp. $\ddot{g}_\alpha$-open, ω-open, g-open, gs-open, gsp-open, α g-open, αgs-open, sg-open, ψ-open, α-open, semi-open) sets is denoted by $\ddot{G}$ O(X) (resp. $\ddot{G}_\alpha$ O(X), ω O(X), G O(X), GS O(X), GSP O(X), αg O(X), αGS O(X), SG O(X), ψ O(X), α O(X), S O(X)).

We denote the power set of X by P(X).

2.4. Definition [17]

A subset S of X is said to be locally closed if S = U ∩ F, where U is open and F is closed in (X, τ).

2.5. Result

(i) Every open set is ψ-open [23].
(ii) Every semi-open set is ψ-open [23].
(iii) Every ψ-open set is sg-open [23].
(iv) Every semi-closed set is sg-closed [7].

3. $\ddot{g}$-Closed Sets

We introduce the following definition.

3.1. Definition

A subset A of X is called a $\ddot{g}$-closed set if cl(A) ⊆ U whenever A ⊆ U and U is sg-open in (X, τ).

3.2. Proposition

Every closed set is $\ddot{g}$-closed.

Proof

If A is any closed set in (X, τ) and G is any sg-open set containing A, then G ⊇ A = cl(A). Hence A is $\ddot{g}$-closed.

The converse of Proposition 3.2 need not be true as seen from the following example.

3.3. Example

Let X = {a, b, c} with τ = {ϕ, {a, b}, X}. Then $\ddot{G}$ C(X) = {ϕ, {c}, {a, c}, {b, c}, X}. Here, A = {a, c} is $\ddot{g}$ -closed set but not closed.

3.4. Proposition

Every $\ddot{g}$ -closed set is $\ddot{g}_\alpha$ -closed.

Proof

If A is a $\ddot{g}$ -closed subset of (X, τ) and G is any sg-open set containing A, then G ⊇ cl(A) ⊇ α cl(A). Hence A is $\ddot{g}_\alpha$ -closed in (X, τ).

The converse of Proposition 3.4 need not be true as seen from the following example.

3.5. Example

Let X = {a, b, c} with τ = {ϕ, {b}, X}. Then $\ddot{G}$ C(X) = {ϕ, {a, c}, X} and $\ddot{G}_\alpha$ C(X) = {ϕ, {a}, {c}, {a, c}, X}. Here, A = {a} is $\ddot{g}_\alpha$ -closed but not $\ddot{g}$ -closed set in (X, τ).

3.6. Proposition

Every $\ddot{g}$ -closed set is ψ -closed.

Proof

If A is a $\ddot{g}$ -closed subset of (X, τ) and G is any sg-open set containing A, then G ⊇ cl(A) ⊇ scl(A). Hence A is ψ -closed in (X, τ).

The converse of Proposition 3.6 need not be true as seen from the following example.

3.7. Example

Let X = {a, b, c} with τ = {ϕ, {a}, X}. Then $\ddot{G}$ C(X) = {ϕ, {b, c}, X} and ψ C(X) = {ϕ, {b}, {c}, {b, c}, X}. Here, A = {b} is ψ -closed but not $\ddot{g}$ -closed set in (X, τ).

3.8. Proposition

Every $\ddot{g}$ -closed set is ω -closed.

Proof

Suppose that A ⊆ G and G is semi-open in (X, τ). Since every semi-open set is sg-open and A is $\ddot{g}$ -closed, therefore cl(A) ⊆ G. Hence A is ω -closed in (X, τ).

The converse of Proposition 3.8 need not be true as seen from the following example.

3.9. Example

Let X = {a, b, c, d} with τ = {ϕ, {d}, {b, c}, {b, c, d}, X}. Then $\ddot{G}$ C(X) = {ϕ, {a}, {a, d}, {a, b, c}, X} and ω C(X) = {ϕ, {a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}, X}. Here, A = {a, c, d} is ω -closed but not $\ddot{g}$ -closed set in (X, τ).

3.10. Proposition

Every ψ -closed set is sg-closed.

Proof

Suppose that A ⊆ G and G is semi-open in (X, τ). Since every semi-open set is sg-open and A is ψ -closed, therefore scl(A) ⊆ G. Hence A is sg-closed in (X, τ).

The converse of Proposition 3.10 need not be true as seen from the following example.

3.11. Example

Let X = {a, b, c} with τ = {ϕ, {a}, {b, c}, X}. Then ψ C(X) = {ϕ, {a}, {b, c}, X} and SG C(X) = P(X). Here, A = {a, b} is sg-closed but not ψ -closed set in (X, τ).

3.12. Proposition

Every ω -closed set is αgs -closed.

Proof

If A is a ω -closed subset of (X, τ) and G is any semi-open set containing A, then G ⊇ cl(A) ⊇ α cl(A). Hence A is αgs -closed in (X, τ).

The converse of Proposition 3.12 need not be true as seen from the following example.

3.13. Example

Let X = {a, b, c} with τ = {ϕ, {a}, X}. Then ω C(X) = {ϕ, {b, c}, X} and αGS C(X) = {ϕ, {b}, {c}, {b, c}, X}. Here, A = {b} is αgs -closed but not ω -closed set in (X, τ).

3.14. Proposition

Every $\ddot{g}$ -closed set is g-closed.

Proof

If A is a $\ddot{g}$-closed subset of (X, τ) and G is any open set containing A, since every open set is sg-open, we have G ⊇ cl(A). Hence A is g-closed in (X, τ).

The converse of Proposition 3.14 need not be true as seen from the following example.

3.15. Example

Let X = {a, b, c} with τ = {ϕ, {a}, {b, c}, X}. Then $\ddot{G}$C(X) = {ϕ, {a}, {b, c}, X} and GC(X) = P(X). Here, A = {a, b} is g-closed but not $\ddot{g}$-closed set in (X, τ).

3.16. Proposition

Every $\ddot{g}$-closed set is αgs-closed.

Proof

If A is a $\ddot{g}$-closed subset of (X, τ) and G is any semi-open set containing A, since every semi-open set is sg-open, we have G ⊇ cl(A) ⊇ αcl(A). Hence A is αgs-closed in (X, τ).

The converse of Proposition 3.16 need not be true as seen from the following example.

3.17. Example

Let X = {a, b, c} with τ = {ϕ, {a}, {b, c}, X}. Then $\ddot{G}$C(X) = {ϕ, {a}, {b, c}, X} and αGSC(X) = P(X). Here, A = {a, c} is αgs-closed but not $\ddot{g}$-closed set in (X, τ).

3.18. Proposition

Every $\ddot{g}$-closed set is αg-closed.

Proof

If A is a $\ddot{g}$-closed subset of (X, τ) and G is any open set containing A, since every open set is sg-open, we have G ⊇ cl(A) ⊇ αcl(A). Hence A is αg-closed in (X, τ).

The converse of Proposition 3.18 need not be true as seen from the following example.

3.19. Example

Let X = {a, b, c} with τ = {ϕ, {c}, {a, b}, X}. Then $\ddot{G}$C(X) = {ϕ, {c}, {a, b}, X} and αgC(X) = P(X). Here, A = {a, c} is αg-closed but not $\ddot{g}$-closed set in (X, τ).

3.20. Proposition

Every $\ddot{g}$-closed set is gs-closed.

Proof

If A is a $\ddot{g}$-closed subset of (X, τ) and G is any open set containing A, since every open set is sg-open, we have G ⊇ cl(A) ⊇ scl(A). Hence A is gs-closed in (X, τ).

The converse of Proposition 3.20 need not be true as seen from the following example.

3.21. Example

Let X = {a, b, c} with τ = {ϕ, {a}, X}. Then $\ddot{G}$C(X) = {ϕ, {b, c}, X} and GSC(X) = {ϕ, {b}, {c}, {a, b}, {a, c}, {b, c}, X}. Here, A = {c} is gs-closed but not $\ddot{g}$-closed set in (X, τ).

3.22. Proposition

Every $\ddot{g}$-closed set is gsp-closed.

Proof

If A is a $\ddot{g}$-closed subset of (X, τ) and G is any open set containing A, every open set is sg-open, we have G ⊇ cl(A) ⊇ spcl(A). Hence A is gsp-closed in (X, τ).

The converse of Proposition 3.22 need not be true as seen from the following example.

3.23. Example

Let X = {a, b, c} with τ = {ϕ, {b}, X}. Then $\ddot{G}$C(X) = {ϕ, {a, c}, X} and GSPC(X) = {ϕ, {a}, {c}, {a, b}, {a, c}, {b, c}, X}. Here, A = {c} is gsp-closed but not $\ddot{g}$-closed set in (X, τ).

3.24. Remark

The following example shows that $\ddot{g}$-closed sets are independent of α-closed sets and semi-closed sets.

3.25. Example

Let X = {a, b, c} with τ = {ϕ, {a, b}, X}. Then $\ddot{G}$C(X) = {ϕ, {c}, {a, c}, {b, c}, X} and αC(X) = SC(X) = {ϕ, {c}, X}. Here, A = {a, c} is

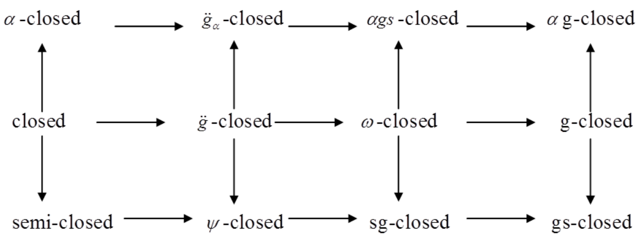
$\ddot{g}$-closed but it is neither α-closed nor semi-closed in (X, τ).

3.26. Example

Let X = {a, b, c} with τ = {ϕ, {a}, X}. Then $\ddot{G}$C(X) = {ϕ, {b, c}, X} and αC(X) = SC(X) = {ϕ, {b}, {c}, {b, c}, X}. Here, A = {b} is α-closed as well as semi-closed in (X, τ) but it is not $\ddot{g}$-closed in (X, τ).

3.27. Remark

From the above discussions and known results in [7, 25, 26, 29, 31], we obtain the following diagram, where A → B (resp. A ◄┼► B) represents A implies B but not conversely (resp. A and B are independent of each other).



None of the above implications is reversible as shown in the remaining examples and in the related papers [7, 25, 26, 29, 31].

4. Properties of $\ddot{g}$-Closed Sets

In this section, we have proved that an arbitrary intersection of $\ddot{g}$-closed sets is $\ddot{g}$-closed. Moreover, we discuss some basic properties of $\ddot{g}$-closed sets.

4.1. Definition

The intersection of all sg-open subsets of (X, τ) containing A is called the sg-kernel of A and denoted by sg-ker(A).

4.2. Lemma

A subset A of (X, τ) is $\ddot{g}$-closed if and only if cl(A) ⊆ sg-ker(A).

Proof

Suppose that A is $\ddot{g}$-closed. Then cl(A) ⊆ U whenever A ⊆ U and U is sg-open. Let x ∈ cl(A). If x ∉ sg-ker(A), then there is a sg-open set U containing A such that x ∉ U. Since U is a sg-open set containing A, we have x ∉ cl(A) and this is a contradiction.

Conversely, let cl(A) ⊆ sg-ker(A). If U is any sg-open set containing A, then cl(A) ⊆ sg-ker(A) ⊆ U. Therefore, A is $\ddot{g}$-closed.

4.3. Proposition

For any subset A of (X, τ), X2 ∩ cl(A) ⊆ sg-ker(A), where X2 = {x ∈ X : {x} is preopen}.

Proof

Let x ∈ X2 ∩ cl(A) and suppose that x ∉ sg-ker(A). Then there is a sg-open set U containing A such that x ∉ U. If F = X − U, then F is sg-closed. Since cl({x}) ⊆ cl(A), we have int(cl({x})) ⊆ A ∪ int(cl(A)). Again since x ∈ X2, we have x ∉ X1 and so int(cl({x})) = ϕ. Therefore, there has to be some y ∈ A ∩ int(cl({x})) and hence y ∈ F ∩ A, a contradiction.

4.4. Theorem

A subset A of (X, τ) is $\ddot{g}$-closed if and only if X1 ∩ cl(A) ⊆ A, where X1 = {x ∈ X : {x} is nowhere dense}.

Proof

Suppose that A is $\ddot{g}$-closed. Let x ∈ X1 ∩ cl(A). Then x ∈ X1 and x ∈ cl(A). Since x ∈ X1, int(cl({x})) = ϕ. Therefore, {x} is semi-closed, since int(cl({x})) ⊆ {x}. Since every semi-closed set is sg-closed [Result 2.5 (4)], {x} is sg-closed. If x ∉ A and if U = X \ {x}, then U is a sg-open set containing A and so cl(A) ⊆ U, a contradiction.

Conversely, suppose that X1 ∩ cl(A) ⊆ A. Then X1 ∩ cl(A) ⊆ sg-ker(A), since A ⊆ sg-ker(A). Now cl(A) = X ∩ cl(A) = (X1 ∪ X2) ∩ cl(A) = (X1 ∩ cl(A)) ∪ (X2 ∩ cl(A)) ⊆ sg-ker(A), since X1 ∩ cl(A) ⊆ sg-ker(A) and Proposition 4.3. Thus, A is $\ddot{g}$-closed by Lemma 4.2.

4.5. Theorem

An arbitrary intersection of $\ddot{g}$-closed sets is $\ddot{g}$-closed.

Proof

Let F = {Ai : i ∈ ∧} be a family of $\ddot{g}$-closed sets and let A = ∩ i ∈ ∧ Ai. Since A ⊆ Ai for each i, X1 ∩ cl(A) ⊆ X1 ∩ cl(Ai) for each i.Using Theorem 4.4 for each $\ddot{g}$-closed set Ai, we have X1 ∩ cl(Ai) ⊆ Ai. Thus, X1 ∩ cl(A) ⊆ X1 ∩ cl(Ai) ⊆ Ai for each i ∈ ∧. That is, X1 ∩ cl(A) ⊆ A and so A is $\ddot{g}$-closed by Theorem 4.4.

4.6. Corollary

If A is a $\ddot{g}$-closed set and F is a closed set, then A ∩ F is a $\ddot{g}$-closed set.

Proof

Since F is closed, it is $\ddot{g}$-closed. Therefore by Theorem 4.5, A ∩ F is also a $\ddot{g}$-closed set.

4.7. Proposition

If A and B are $\ddot{g}$-closed sets in (X, τ), then A ∪ B is $\ddot{g}$-closed in (X, τ).

Proof

If A ∪ B ⊆ G and G is sg-open, then A ⊆ G and B ⊆ G. Since A and B are $\ddot{g}$-closed, G ⊇ cl(A) and G ⊇ cl(B) and hence G ⊇ cl(A) ∪ cl(B) = cl(A ∪ B). Thus A ∪ B is $\ddot{g}$-closed set in (X, τ).

4.8. Proposition

If a set A is $\ddot{g}$-closed in (X, τ), then cl(A) – A contains no nonempty closed set in (X, τ).

Proof

Suppose that A is $\ddot{g}$-closed. Let F be a closed subset of cl(A) – A. Then A ⊆ Fc. But A is $\ddot{g}$-closed, therefore cl(A) ⊆ Fc. Consequently, F ⊆ (cl(A))c. We already have F ⊆ cl(A). Thus F ⊆ cl(A) ∩ (cl(A))c and F is empty.

The converse of Proposition 4.8 need not be true as seen from the following example.

4.9. Example

Let X = {a, b, c} with τ = {ϕ, {a}, X}. Then $\ddot{G}$C(X) = {ϕ, {b, c}, X}. If A = {b}, then cl(A) – A = {c} does not contain any nonempty closed set. But A is not $\ddot{g}$-closed in (X, τ).

4.10. Theorem

A set A is $\ddot{g}$-closed if and only if cl(A) – A contains no nonempty sg-closed set.

Proof

Necessity. Suppose that A is $\ddot{g}$-closed. Let S be a sg-closed subset of cl(A) – A. Then A ⊆ Sc. Since A is $\ddot{g}$-closed, we have cl(A) ⊆ Sc. Consequently, S ⊆ (cl(A))c. Hence, S ⊆ cl(A) ∩ (cl(A))c = ϕ. Therefore S is empty.

Sufficiency. Suppose that cl(A) – A contains no nonempty sg-closed set. Let A ⊆ G and G be sg-open. If cl(A) ⊄ G, then cl(A) ∩ Gc ≠ ϕ. Since cl(A) is a closed set and Gc is a sg-closed set, cl(A) ∩ Gc is a nonempty sg-closed subset of cl(A) – A. This is a contradiction. Therefore, cl(A) ⊆ G and hence A is $\ddot{g}$-closed.

4.11. Proposition

If A is $\ddot{g}$-closed in (X, τ) and A ⊆ B ⊆ cl(A), then B is $\ddot{g}$-closed in (X, τ).

Proof

Since B ⊆ cl(A), we have cl(B) ⊆ cl(A). Then, cl(B) – B ⊆ cl(A) – A. Since cl(A) – A has no nonempty sg-closed subsets, neither does cl(B) – B. By Theorem 4.10, B is $\ddot{g}$-closed.

4.12. Proposition

Let A ⊆ Y ⊆ X and suppose that A is $\ddot{g}$-closed in (X, τ). Then A is $\ddot{g}$-closed relative to Y.

Proof

Let A ⊆ Y ∩ G, where G is sg-open in (X, τ). Then A ⊆ G and hence cl(A) ⊆ G. This implies that Y ∩ cl(A) ⊆ Y ∩ G. Thus A is $\ddot{g}$-closed relative to Y.

4.13. Proposition

If A is a sg-open and $\ddot{g}$-closed in (X, τ), then A is closed in (X, τ).

Proof

Since A is sg-open and $\ddot{g}$-closed, cl(A) ⊆ A and hence A is closed in (X, τ).

Recall that a topological space (X, τ) is called extremally disconnected if cl(U) is open for each U ∈ τ.

4.14. Theorem

Let (X, τ) be extremally disconnected and A a semi-open subset of X. Then A is $\ddot{g}$-closed if and only if it is sg-closed.

Proof

It follows from the fact that if (X, τ) is extremally disconnected and A is a semi-open subset of X, then scl(A) = cl(A) (Lemma 0.3 [15]).

4.15. Theorem

Let A be a locally closed set of (X, τ). Then A is closed if and only if A is $\ddot{g}$-closed.

Proof

(i) ⇒ (ii). It is fact that every closed set is $\ddot{g}$-closed.

(ii) ⇒ (i). By Proposition 5.1.3.3 of Bourbaki [6], A ∪ (X – cl(A)) is open in (X, τ), since A is locally closed. Now A ∪ (X – cl(A)) is sg-open set of (X, τ) such that A ⊆ A ∪ (X – cl(A)). Since A is $\ddot{g}$-closed, then cl(A) ⊆ A ∪ (X – cl(A)). Thus, we have cl(A) ⊆ A and hence A is a closed.

4.16. Proposition

For each x ∈ X, either {x} is sg-closed or {x}c is $\ddot{g}$-closed in (X, τ).

Proof

Suppose that {x} is not sg-closed in (X, τ). Then {x}c is not sg-open and the only sg-open set containing {x}c is the space X itself. Therefore cl({x}c) ⊆ X and so {x}c is $\ddot{g}$-closed in (X, τ).

4.17. Theorem

Let A be a $\ddot{g}$-closed set of a topological space (X, τ). Then,

(i) sint(A) is $\ddot{g}$-closed.

(ii) If A is regular open, then pint(A) and scl(A) are also $\ddot{g}$-closed sets.

(iii) If A is regular closed, then pcl(A) is also $\ddot{g}$-closed.

Proof

(i) Since cl(int(A)) is a closed set in (X, τ), by Corollary 4.6, sint(A) = A ∩ cl(int(A)) is $\ddot{g}$-closed in (X, τ).

(ii) Since A is regular open in X, A = int(cl(A)). Then scl(A) = A ∪ int(cl(A)) = A. Thus, scl(A) is $\ddot{g}$-closed in (X, τ). Since pint(A) = A ∩ int(cl(A)) = A, pint(A) is $\ddot{g}$-closed.

(iii) Since A is regular closed in X, A = cl(int(A)). Then pcl(A) = A ∪ cl(int(A)) = A. Thus, pcl(A) is $\ddot{g}$-closed in (X, τ).

The converses of the statements in the Theorem 4.17 are not true as we see in the following examples.

4.18. Example

Let X = {a, b, c} with τ = {ϕ, {c}, {b, c}, X}. Then $\ddot{G}$C(X) = {ϕ, {a}, {a, b}, X}. Then the set A = {b} is not a $\ddot{g}$-closed set. However sint(A) = ϕ is a $\ddot{g}$-closed.

4.19. Example

Let X = {a, b, c} with τ = {ϕ, {a}, {b}, {a, b}, {a, c}, X}. Then $\ddot{G}$C(X) = {ϕ, {b}, {c}, {a, c}, {b, c}, X}. Then the set A = {c} is not regular open.

However A is $\ddot{g}$-closed and scl(A) = {c} is a $\ddot{g}$-closed and pint(A) = ϕ is also $\ddot{g}$-closed.

4.20. Example

Let X = {a, b, c} with τ = {ϕ, {a, b}, X}. Then $\ddot{G}$C(X) = {ϕ, {c}, {a, c}, {b, c}, X}. Then the set A = {c} is not regular closed. However A is a $\ddot{g}$-closed and pcl(A) = {c} is $\ddot{g}$-closed.

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